

CONTROLLED QUANTUM PACKETS

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Abstract

We build Quantum wave packets as dynamically controlled systems. It is useful to use, to this aim, Stochastic Mechanics, a probabilistic simulation of Quantum Mechanics.

1 Introduction

We look at time evolution of a physical system from the point of view of dynamical control theory. Normally we solve motion equation with a given external potential and we obtain time evolution. Standard examples are the trajectories in classical mechanics or the wave functions in Quantum Mechanics. In the control theory, we have the configurational variables of a physical system, we choose a velocity field and with a suited strategy we force the physical system to have a well defined evolution. The evolution of the system is the "premium" that the controller receives if he has adopted the right strategy. The strategy is given by well suited laboratory devices. The control mechanisms are in many cases non linear; it is necessary, namely, a feedback mechanism to retain in time the selected evolution. Our aim is to introduce a scheme to obtain Quantum wave packets by control theory. The program is to choose the characteristics of a packet, that is, the equation of evolution for its centre and a controlled dispersion, and to give a building scheme from some initial state (for example a solution of stationary Schroedinger equation). It seems natural in this view to use stochastic approach to Quantum Mechanics, that is, Stochastic Mechanics [S.M.] [2] [3]. It is a quantization scheme different from ordinary ones only formally. This approach introduces in quantum theory the whole mathematical apparatus of stochastic control theory. Stochastic Mechanics, in our view, is more intuitive when we want to study all the classical-like problems. We apply our scheme to build two classes of quantum packets both derived generalizing some properties of coherent states [4].

2 Stochastic Mechanics

We give a brief outline of S.M.. A way to introduce S.M. can be the following. We consider a diffusion process with diffusion coefficient $\nu(q, t)$. $q(t)$ is a stochastic dynamical variable. We introduce its associated Ito forward and backward equations

$$dq(t) = v_{(+)}(q(t), t)dt + \nu(q, t)dw(t), \quad dt > 0. \quad (1)$$

$$dq(t) = v_{(-)}(q(t), t)dt + \nu(q, t)dw(t), \quad ; dt > 0. \quad (2)$$

In the above stochastic differential equations $v_{(+)}$ and $v_{(-)}$ are respectively forward and backward drift fields, and w is the Wiener process. We can equivalently consider forward Fokker Planck equation

$$\partial_t \rho(q, t) = \nu \Delta \rho(q, t) - \nabla v_{(+)} \rho \quad (3)$$

and the correspondent backward one with $v_{(-)}$. For $q(t)$ are also defined the osmotic velocity

$$u(x, t) = \frac{v_{(+)} - v_{(-)}}{2}, \quad (4)$$

and the current velocity

$$v(x, t) = \frac{v_{(+)} + v_{(-)}}{2}. \quad (5)$$

The simple identity holds

$$v_{(+)} = v_{(-)} + \frac{\nu \nabla \rho}{\rho}. \quad (6)$$

The sum of the Fokker-Planck equations, using (5), gives us the continuity equation:

$$\partial_t \rho = -\nabla(\rho v), \quad (7)$$

thus expressing probability storage. Now if we assign "a priori" $v_{(+)}$ or $v_{(-)}$, the diffusion process so introduced is completely determined. By the integration of Ito equations (equivalently of Fokker Planck equations) we have the complete evolution of the dynamical stochastic system under study. A notable [5] inequality to take in account is the following

$$\Delta q \Delta u \geq \langle \nu(q, t) \rangle. \quad (8)$$

It derives by the nondifferentiability of the process. We remark that diffusion processes are not time reversal invariant. There is, however, a time reversal approach to diffusion processes, thus introducing a very different class of diffusions. In this different way Ito equations become a kinematical condition to complete with a suitable dynamical principle. It comes simple to add as a dynamical condition a variational principle. Namely, choosing the following mean regularized Eulerian action A:

$$A = \int_{t_a}^{t_b} \left[\frac{m}{2} (v^2 - u^2) - \Phi \right] \rho d^3 x, \quad (9)$$

where $\Phi(x, t)$ denotes the external potential, taking smooth variations, with the continuity equation (7) taken as a constraint, after standard calculations we obtain Hamilton-Jacobi-Madelung (H-J-M) equation

$$\partial_t v + (v \cdot \nabla) v - \nu^2 \nabla \left(\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right) = -\nabla \Phi. \quad (10)$$

The current velocity is fixed to be a gradient field $v = \nabla S/m$, with $S(x, t)$ a scalar field. This class of diffusion processes have time reversal invariance and is commonly appelled "Stochastic Mechanics". S.M., in fact, presents us as a generalization of classical mechanics in which ordinary classical trajectories become probabilistic. There are many applications of S.M. (biological population segregation, bode law, planetary atmosphere, stochastic neurodynamics); it is then a theory interesting for itself. It is well known that the equations of S.M. show, also, some interesting link with the equations of Quantum Mechanics.

3 Quantum Mechanics and Stochastic Mechanics

Now we introduce an interpretative scheme in which phenomenological previsions of S.M. coincide with that of Quantum Mechanics for all experimentally measurable quantum effects. S.M., in this view, is a quantization scheme different from ordinary ones only formally, but completely equivalent from the point of view of physical interpretation. Stochastic Mechanics can be interpreted as a probabilistic simulation of Quantum Mechanics giving a bridge between Quantum Mechanics and stochastic differential calculus. Defining now the complex function $\Psi = \sqrt{\rho} \exp [iS/\hbar]$, where ρ and S are the same that satisfy the equations of Stochastic Mechanics, and choosing $\nu = \frac{\hbar}{2m}$, we immediately have that the continuity equation (7) together with the dynamical equation (8) are equivalent to the Schrödinger equation. The correspondence between expectations and correlations defined in the stochastic and in the canonic pictures are

$$\langle \hat{q} \rangle = E(q), \quad \langle \hat{p} \rangle = mE(v), \quad (11)$$

$$\Delta \hat{q} = \Delta q, \quad (\Delta \hat{p})^2 = m^2[(\Delta u)^2 + (\Delta v)^2].$$

The following chain inequality holds:

$$(\Delta \hat{q})^2 (\Delta \hat{p})^2 \geq m^2 (\Delta q)^2 (\Delta u)^2 \geq \frac{\hbar^2}{4}. \quad (12)$$

In the above relations \hat{q} and \hat{p} denote the position and momentum observables in the Schrödinger picture, $\langle \cdot \rangle$ denotes the expectation value of the operators in the given state Ψ , $E(\cdot)$ is the expectation value of the stochastic variables associated in the Nelson picture to the state $\{\rho, v\}$, and $\Delta(\cdot)$ denotes the root mean square deviation.

The chain of inequalities (11), i.e. the osmotic uncertainty relation and its equivalence with the momentum-position uncertainty, were proven in Ref. [6].

4 Ehrenfest equation and quantum packets

It is opportune to give a brief outline of the standard arguments about wave packets motion. It is usual to start with the Ehrenfest equation. The argument is the following. In order to have a wave packet following a controlled motion, that is the packet motion may be likened to the motion of classical particle, it is not only necessary that its position and its momentum follow the laws of classical mechanics, but we must control the dimensions of the packet; it must remain small or controlled at any time. In fact (we pose $m = 1, \hbar = 1/2$), if we look at Ehrenfest equations

$$\frac{d}{dt} E(q) = E(v) \quad (13)$$

and

$$\frac{d^2}{dt^2} E(q) = -E(\nabla \Phi), \quad (14)$$

(written directly in the stochastic formalism), it is immediately seen that all the moments of ρ are implicated through the mean values. We can see this in an intuitive manner. Consider the

motion of a particle in an external potential Φ . In order to have a classical-like motion the "right" Ehrenfest equation should be

$$\frac{d^2}{dt^2}E(q) = -\nabla\Phi|_{E(q)}. \quad (15)$$

If we take Taylor expansion of $\nabla\Phi$, all the moments are contained in the Ehrenfest equation as a matter of fact. Now it is necessary to obtain a set of equations that rule the evolution of moments. It is not difficult to see that it is satisfied the following set of equations

$$\frac{1}{n} \frac{d}{dt} E((q - E(q))^n) = E((q - E(q))^{n-1}v) - E((q - E(q))^{n-1})E(v). \quad (16)$$

They are interesting by itself. From these equations it is possible to connect moments and the external potential. This set of equations has in general a very complicated structure, namely, we have infinite coupled equations, and only in some particular case the equations collapse to a finite number (for example when ρ has some gaussian behaviour, as in our examples). The equations express the fact that the positional dispersion is controlled by the whole density. The equation to consider, in general, is the equation for positional Entropy

$$\frac{d}{dt} E(\log \rho) = -E(\nabla v). \quad (17)$$

5 Controlled quantum packets

Now we illustrate the scheme to build controlled quantum systems. We prove that it is theoretically possible, choosing a well suited control device, to have packets with a well defined motion and form. This point of view is not new; namely the coherent states and in particular the parametric oscillator are, in an opportune sense, linearly controlled systems in which the equation for dispersion depend by the coefficients of the external potential. The general scheme of control we introduce is non-linear. Our idea is the following. If we select a particular current velocity, we choose, in fact, the phase of the wave function and, as a consequence, we choose the characteristics of the motion of the centre of the packet. Moreover, a choice of current velocity selects a class of solutions of continuity (Fokker-Planck) equation. The(H-J-M) equation becomes in this scheme a constraint to retain time-reversal invariance, giving us the controlling device. Now we build two classes of controlled quantum packets as example. We need some initial condition ρ_0 for probability density; it can be a generic L^1 function and we can choose always that it satisfies a stationary Schrödinger equation with Φ_0 as external potential. In the first example we take the current velocity selected as that of coherent states [7]

$$v = E(v) + \frac{x - E(q)}{\Delta q} \frac{d\Delta q}{dt}. \quad (18)$$

As second example we impose gaussian behaviour simply balancing current and osmotic velocity :

$$v = E(v) - \frac{1}{2} u(x, t) \frac{d}{dt} \Delta q. \quad (19)$$

6 Generalized coherent states

We have already introduced the first example as generalized coherent states [7] [8]. If we insert the current velocity in continuity equation (7) we solve in a very simple way and we obtain:

$$\rho(\xi) = \int \delta(y - \xi) \rho_0(y) dy, \quad \xi = \frac{x - E(q)}{\Delta q}. \quad (20)$$

Now we can examine the Ehrenfest equation, and then (H-J-M) equation that now is become an identity. It is not difficult to see that

$$\frac{dE(v)}{dt} x + \frac{1}{2} \xi^2 \frac{d^2 \Delta q}{dt^2} - \frac{\int \Phi_0(y) \rho_0(y) \delta(y - \xi) dy}{\int \rho_0(y) \delta(y - \xi) dy} + L(t) = -\Phi. \quad (21)$$

Φ_0 is the external potential in the stationary Schrödinger equation of which ρ_0 is a solution, Φ is the state dependent control device. Inserting now our current velocity (19) in the eq.(17) we see that

$$E(\log \rho(x, t)) = -\log \Delta q. \quad (22)$$

The whole positional entropy come by dispersion and this means that the set of eq.s (16) is close. We can extract from eq.s (18)-(20) one equation for Δq , and all the others depend from this last one. The equation is:

$$\frac{d^2 \Delta q}{dt^2} = \frac{a}{\Delta q^3} - E(\xi \nabla \Phi) \quad (23)$$

where a is a number. The Ehrenfest equation becomes for this states

$$\frac{d^2}{dt^2} E(q) = -\nabla \Phi|_{E(q)}. \quad (24)$$

The couple of equations (24), (25) comes from equation (22) taking the first and second order of Taylor expansion. It is, also, significative to write Ito equation for this class of stochastic processes

$$dq(t) = E(v)dt + \nu dw. \quad (25)$$

They are associated, as Glauber states, to Wiener processes with a drift that is solution of the classical Eherenfest equation (25). For more details see [9].

7 Controlled gaussian wave packets

Now we give our second example [10]. If we insert the current velocity (22) into the continuity equation (7) we have the following Fokker-Planck equation

$$\partial_t \rho(x, t) = E(q) \nabla \rho(x, t) + \frac{d}{dt} \Delta q \nabla^2 \rho(x, t), \quad (26)$$

whose general solution is

$$\rho(\xi, t) = \frac{1}{\Delta q} \int \rho_0(y) \exp -[(x - \xi)^2] dy. \quad (27)$$

Now, also in this case it is possible to verify that Ehrenfest equation is classical-like; it sufficient to control the first and second moment. The equation for positional entropy is very simple as in the first example, and we have the following control potential:

$$\frac{dE(v)}{dt}x - \log \rho(x, t) \frac{d^2}{dt^2} \Delta q^2 - (1 + (\frac{d \Delta q^2}{dt} \nu)^2) [\frac{\int \Phi_0(y) G(y, \xi) dy}{\int G(y, \xi) dy} + \frac{\int u_0(y)^2 G(y, \xi) dy}{\int G(y, \xi)}] - [\frac{\int u_0(y) G(y, \xi)}{\int G(y, \xi) dy}]^2 + N(t) = -\Phi \quad (28)$$

where $G(y, \xi) = \rho_0(y) \exp[(y - \xi)^2]$, and $N(t)$ is a generic time function. Also in this case the Ehrenfest equation is classical like. It is not difficult to see that

$$E(\nabla \Phi) - \nabla \Phi|_{E(q)} = F(E(q), \Delta q). \quad (29)$$

Using this identity and expanding (28), by the first and second order one obtain Ehrenfest equation and the coupled equation for dispersion. Note that the Ito equation is now

$$dq(t) = E(v)dt + \nu \frac{d\Delta q}{dt} dw. \quad (30)$$

This wave packets are Gaussian modulation of the initial state. The equations are all implicit if we do not specify the initial condition.

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